## Title

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# PERFORMANCE ANALYSIS OF A BROADCAST STAR LOCAL AREA NETWORK WITH COLLISION AVOIDANCE:
## Part 2, Finite Station Population Model

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# Performance Analysis of a Broadcast Star Local Area Network with Collision Avoidance: Part 2, Finite Station Population Model*

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**Abstract**

In order to overcome the performance bottleneck due to collisions and their resolution in random access LANs, a new network architecture called a broadcast star network with collision avoidance has been proposed and studied by many researchers [2 - 15]. The performance of a broadcast star network is also studied in a companion paper [11], assuming an infinite station population and synchronous operation of the network.

In this paper, we analyze the performance of a broadcast star network assuming a finite number of stations, and we obtain the throughput and the distribution of transmission delays. An exact analysis is presented first. However, this analysis is computationally practical for networks with a small number of stations only. Therefore, an approximate analysis is also presented for a network having a large number of stations. The accuracy of our approximation is examined through simulations.

## 1. Introduction

Random access protocols exhibit small transmission delays under light traffic conditions, since stations transmit as soon as they want access to the channel, and the probability of a collision is low when traffic is light. However, as the network traffic increases, so do the chances for collisions. The result is that channel utilization is reduced and packet delays are increased [1].

In order to resolve this performance bottleneck in random access LANs, the broadcast star network with collision avoidance has been proposed and studied [2 - 15]. An experimental broadcast star network can be found in [4, 5]. Papers [6 - 8, 14, 15] discuss various station and switch protocols for networks based on collision avoidance. The performance has been studied in [9 - 13].

In a broadcast star network, stations are connected to a central collision avoidance switch by full duplex channels. Each of these channels comprises an uplink and a downlink. The channel time is divided into slots, and stations transmit only at the beginning of a slot*.

The broadcast star network is similar to ALOHA networks in that two or more stations may transmit packets at the same time. However, the broadcast star avoids losing the use of the channel since collisions of simultaneously transmitted packets are eliminated by a collision avoidance switch. If two or more packets arrive at the switch simultaneously, it randomly selects one of the packets coming from the uplinks and blocks (or discards) all

* There are two possible operations of the broadcast star network: synchronous and asynchronous operations. In synchronous operation, the channel time is divided into slots, and stations transmit only at the beginning of a slot. In asynchronous operation, transmissions of packets are not confined to the beginning of time slots, and stations may start transmission any time. As shown in a companion paper [11], a broadcast star operating under synchronous mode yields better performance (i.e., smaller transmission delay and higher throughput) than its asynchronous counterpart. Therefore we will only consider synchronous operation of the network in this paper.

the other packets. The switch then broadcasts the selected packet on all the downlinks. It should be noted that a collision avoidance switch does not buffer packets; no memory is used to store-and-forward packets.

The switch has two states. It is busy from the time it has selected a packet to the time it has finished broadcasting the packet on the downlinks. Otherwise the switch is idle. While the switch is busy, all packets arriving on the uplinks are ignored in their entirety. Upon going idle, the switch randomly selects one of the newly arriving packets to broadcast.

The important feature of a collision avoidance switch is that when two or more packets contend for a switch, it is guaranteed that one of the packets acquires the switch and is successfully transmitted through it. Thus, no channel time is wasted in the transmission of collided packets, and the traditional penalty of random access is eliminated. Collision avoidance can be implemented with very little circuitry. Implementation examples of a collision avoidance switch are given in [2 - 4, 9].

The station protocol for the slotted broadcast star is very simple and is like slotted ALOHA:

(1) A station transmits a newly arrived packet in the next slot following the time of its arrival.

(2) After a propagation delay to and from the switch, the station monitors its downlink for the broadcast of its packet.

(3) If the station does not see the start of its packet, then it retransmits the packet immediately in the following slot.

(4) If the station does see its packet, it knows that the packet has won the switch and will be broadcast in its entirety.

Note that in a broadcast star, stations do not defer retransmission. Stations resubmit their packets as soon as they learn of transmission failure. Since no collision with an ongoing transmission can occur in this broadcast star network, there is no need for the random delays that ALOHA requires.

In a companion paper [11], the performance of a broadcast star network has been analyzed assuming an infinite number of stations. In this paper, we assume a finite number of stations and obtain the throughput and the distribution of transmission delays. In section 2, our assumptions and notations are described. In section 3, an exact analysis is presented. However, this analysis is computationally practical for networks with a small number of stations only. Therefore, in section 4, an approximate analysis is presented for a network having a large number of stations. In section 5, numerical results are shown, and the accuracy of our approximation is examined through simulations.

## 2. Model, Notations and Assumptions

We assume constant length packets. Time is slotted, and the slot length is equal to a packet transmission time (packet length in bits divided by the channel speed). We use the slot length as a unit time.

We let $N$ be the number of stations on the network. Stations are assumed to be homogeneous, that is, the statistical characteristics of packet arrivals at different stations are independent and identical. The arrival of a new packet occurs at a station with the probability $q$ (per slot). Each station has a buffer to store only one packet. We assume that a station keeps a copy of a packet in its buffer until the packet is successfully received by its destination. More precisely, we assume that the packet is kept in the buffer until the end of the slot where the station starts receiving its successful broadcast. When a station has a packet in its buffer, it is called busy. Otherwise, it is called idle. A packet arriving at a busy station will be lost.

Stations are located at the same distance from the switch. Let $R$ denote the round trip propagation delay to and from the switch (measured in slots). Since it takes $R$ (slots) for a station to know whether a transmission is successful or not, and since blocked packets are immediately retransmitted in the following slot, the time required for a retransmission is $\lfloor R \rfloor + 1$ (slots). Here, $\lfloor R \rfloor$ denotes the largest integer less than or equal to $R$. When a new packet is accepted at a station, it is immediately transmitted in the slot following its arrival. If an initial transmission fails, a station retransmits a packet in every $\lfloor R \rfloor + 1$-st subsequent slot until the packet is successfully transmitted. Thus, transmissions of a packet are confined in this subsequence of slots which begins with the slot following the arrival of a packet and continues with every $\lfloor R \rfloor + 1$-st slot thereafter. See Figure 1. Therefore, we can partition the slots into $\lfloor R \rfloor + 1$ subsequences, each of which is called a logical channel or just a channel for short. Let $L$ denote the number of channels, i.e., $L = \lfloor R \rfloor + 1$. Thus slots $1, 2, \ldots, \lfloor R \rfloor, \lfloor R \rfloor + 1, \lfloor R \rfloor + 2, \lfloor R \rfloor + 3, \ldots$ belong to channels $1, 2, \ldots \lfloor R \rfloor, L, 1, 2, \ldots$, respectively. Let $s_i$ denote a slot which belongs to channel $i$. A series of $L$ consecutive slots, $\{s_1, s_2, \cdots, s_L\}$, constitute a cycle. The number of stations transmitting in a channel is called the population of the channel.

We define the transmission delay $D$ as the time (measured in slots) from the beginning of the slot immediately following the arrival of a packet to the time at which the packet is completely received at its destination. If $m$ retransmissions are required for a packet to be successfully transmitted, transmission delay $D$ is the sum of the time required for $m$ retransmissions and the time for a successful transmission. One retransmission takes $\lfloor R \rfloor + 1$ (slots), and a successful transmission takes $R + 1$ (slots). Therefore, the transmission delay $D$ (slots) for a packet which is successfully transmitted after $m$ ($m \geq 0$) retransmissions is given by:

$$D = m \times (\lfloor R \rfloor + 1) + R + 1. \qquad (2-1)$$

From eq.(2-1), the mean and the variance of $D$ become:

$$E[D] = E[m](\lfloor R \rfloor + 1) + R + 1, \qquad (2-2)$$

$$Var[D] = Var[m](\lfloor R \rfloor + 1)^2. \qquad (2-3)$$

Higher moments of $D$ can be also obtained from eq.(2-1).

In the following, we obtain the statistical characteristics of the number of retransmissions $m$ required for a packet to be successfully transmitted, the only unknown factor in eq.(2-1). The number of required retransmissions $m$ depends on the number of stations attempting to access the same channel. Although transmissions in different channels never contend with one another, they are not totally independent. The population of one channel depends on the population of other channels. Therefore, in order to obtain $m$, the population of each of the channels needs to be observed.

## 3. Exact Analysis of a Broadcast Star Network with a Finite Station Population

In this section, we present an exact analysis for the performance of a broadcast star network assuming a finite number ($N$) of stations. The throughput, the mean, and the variance of transmission delay are obtained. The round trip delay $R$ is assumed to be greater than or equal to 1.0 in this section. For the case $R \leq 1$, the analysis presented in section 4 gives exact results.

Since we assume a homogeneous network, channels are statistically identical in steady state. Therefore, we observe channel 1 and obtain the performance measures there without loss of generality. Our analysis is based on an imbedded Markov chain. We observe the network in slots belonging to channel 1 (imbedded points), and obtain a set of linear equations relating system states at successive imbedded points. In the example shown in Figure 1, slots in channel 1 are indicated by a circle.

### 3.1 State of a Station and System State

In a broadcast star network, stations are in one of the following states:

- Idle State (denoted by **ID**), where a station is idle and has no packet to send.
- Initial Transmission State (denoted by **IT**), where a station is transmitting a new packet that had arrived during the previous slot.
- Retransmission State (denoted by $(\mathbf{RT}, i)$), where a station has a packet to be retransmitted. $i$ ($i = 1, 2, \ldots, L$) refers to the channel in which the packet is initially transmitted. If an initial transmission fails, a station immediately goes into this state and stays in this state until the successful transmission of the packet takes place.
- Successful Transmission State (denoted by $(\mathbf{ST}, i)$), where a station has sent a packet successfully and is waiting for a broadcast of its packet. $i$ ($i = 1, 2, \ldots, L$) refers to the channel in which the packet is successfully transmitted. A station goes into this state immediately after the successful transmission of a packet in channel $i$ (slot $s_i$), and stays in this state until the end of the broadcast of its packet.

To illustrate state transitions at a station, an example is given below. Note that, as explained in section 2, slots are partitioned into cycles, each consisting of slots $s_1$, $s_2$, $\cdots$, $s_L$, where $s_i$ denotes a slot belonging to channel $i$. Assume a new packet arrives at a station during slot $s_{i-1}$. Initial transmission of this packet takes place in the next slot $s_i$, and the station goes into the initial transmission state **IT** in $s_i$. Assume that this initial transmission fails. The station consequently goes into a retransmission state $(\mathbf{RT}, i)$ in $s_{i+1}$, immediately after the initial transmission. $i$ of $(\mathbf{RT}, i)$ indicates the channel where the initial transmission of the packet took place. If the propagation delay $R$ is large, the station may not know the outcome of its transmission attempt in $s_i$ by the beginning of slot $s_{i+1}$. Nevertheless, we assume that the station goes into a retransmission state $(\mathbf{RT}, i)$ in slot $s_{i+1}$, if the transmission is bound to fail. This station stays in a retransmission state $(\mathbf{RT}, i)$ until the retransmission of the packet takes place in slot $s_i$ of the next cycle. The station is still in a retransmission state $(\mathbf{RT}, i)$ in slot $s_i$ where a retransmission is taking place. Now, assume that this retransmission is successful. The station then moves to a successful transmission state $(\mathbf{ST}, i)$ in slot $s_{i+1}$, immediately after the retransmission. (If the propagation delay $R$ is large, the station may not know the outcome of its transmission attempt in $s_i$ by the beginning of slot $s_{i+1}$. Nevertheless, we assume that the station goes into a successful transmission state in slot $s_{i+1}$, if the retransmission is bound to succeed.) The station remains in a successful transmission state until it receives a complete broadcast of its packet. More precisely, the station remains in a successful transmission state until the end of the slot during which the successful broadcast is completely received. Assume that there is no new packet arrival during this slot, then the station goes into an idle state **ID** in the next slot. Observe that a station in state $(\mathbf{RT}, i)$ or $(\mathbf{ST}, i)$ may change its state only in slot $s_i$.

Let $e_k$ represent the state of station $k$. Then $e_k$ is one of the following $2 \times L + 2$ possible states: **ID**, **IT**, $(\mathbf{RT}, i)$, or $(\mathbf{ST}, i)$; where $i$ ($i = 1, 2, \ldots, L$) indicates the channel that the initial transmission of a packet took place. If there are $N$ stations on a network, a state of the network can be represented by a tuple $(e_1, e_2, \cdots, e_N)$. The number of possible system states is quite large and is on the order of $(2 \times L + 2)^N$. Due to the large number of possible system states, the exact analysis presented in this section is computationally practical only for networks with a small number of stations. We arbitrarily enumerate all the possible system states, and let $\mathbf{E}_j$ represent the $j$-th element. Note that $1 \le j \le n_{state}$, where $n_{state}$ is the number of possible system states. Accordingly, elements of $\mathbf{E}_j$ are given suffix $j$:

$$\mathbf{E}_j = (e_1^j, e_2^j, e_3^j, \ldots, e_N^j) \qquad (1 \le j \le n_{state}), \qquad (3-1)$$

where $e_k^j$ $(k = 1, 2, \ldots, N)$ represents the state of station $k$ in system state $\mathbf{E}_j$.

### 3.2 State Transition Probabilities

We first obtain state transition probabilities between successive slots. From the slot-by-slot probabilities, the state transition probabilities between imbedded points are obtained.

In the following, we observe the system in two consecutive slots $s_i$ and $s_{i+1}$. Note that if $i = L$, the slot next to $s_L$ is $s_1$ in the next cycle, not $s_{L+1}$. However, in order to avoid unnecessary complication, we simply use $s_i$ and $s_{i+1}$ to represent two consecutive slots.

Let's first consider the possible state transitions at a station. The only possible transition from an idle state **ID** in slot $s_i$ is to an initial transmission state **IT** in slot $s_{i+1}$. The arrival of a new packet at a station during slot $s_i$ causes this transition.

If a station is in an initial transmission state **IT** during $s_i$, it moves either to a successful transmission state $(\mathbf{ST}, i)$ if the transmission is successful, or to a retransmission state $(\mathbf{RT}, i)$ during slot $s_{i+1}$ otherwise.

If a station is in a retransmission state during slot $s_i$, that station is either retransmitting a packet in $s_i$ (if the state is $(\mathbf{RT}, i)$) or waiting for feedback of its previous transmission attempt in channel $j$ (if the state is $(\mathbf{RT}, j)$, $j \neq i$). If the station is in state $(\mathbf{RT}, i)$ during slot $s_i$, then in the next slot, $s_{i+1}$, the station either moves to state $(\mathbf{ST}, i)$ if the retransmission is successful, or remains in the same state $(\mathbf{RT}, i)$ otherwise. If the station is in state $(\mathbf{RT}, j)$ $(j \neq i)$ during slot $s_i$, it remains in the same state $(\mathbf{RT}, j)$ during slot $s_{i+1}$.

If a station is in a successful transmission state during slot $s_i$, that station is either waiting for a broadcast of its packet (if the state is $(\mathbf{ST}, j)$ $j \neq i$) or actually receiving a broadcast of its packet (if the state is $(\mathbf{ST}, i)$). If the station is in state $(\mathbf{ST}, j)$ $(j \neq i)$ during slot $s_i$, (the station is waiting for a broadcast of its packet), it remains in the same state $(\mathbf{ST}, j)$ in the next slot $s_{i+1}$. If the station is in state $(\mathbf{ST}, i)$ during slot $s_i$ (the station is actually receiving a broadcast of its packet in slot $s_i$), it either moves to an idle state **ID**, if there is no arrival at the station during slot $s_i$, or moves to a transmission state **IT**, if there is a new packet arrival during slot $s_i$.

The transitions listed above are the only possible transitions at a station. From the above observation, we can obtain the state transition probabilities from slot $s_i$ to slot $s_{i+1}$. Let $p^i_{\mathbf{E}_j,\mathbf{E}_k} = \Pr[\mathbf{E}_k \text{ in slot } s_{i+1} \mid \mathbf{E}_j \text{ in slot } s_i]$ denote the one step state transition probability from state $\mathbf{E}_j$ in slot $s_i$ to state $\mathbf{E}_k$ in slot $s_{i+1}$.

If a transition from $\mathbf{E}_j = (e^j_1, e^j_2, e^j_3, \ldots, e^j_N)$ to $\mathbf{E}_k = (e^k_1, e^k_2, e^k_3, \ldots, e^k_N)$ requires a transition (or transitions) other than those discussed above, transition from $\mathbf{E}_j$ to $\mathbf{E}_k$ is not possible, thus the corresponding transition probability $p^i_{\mathbf{E}_j,\mathbf{E}_k}$ is zero.

For the other cases (i.e., for the cases of possible system state transitions), the transition probabilities are obtained in the following way. Assume that the system is in state $\mathbf{E}_j$ in slot $s_i$ and moves to state $\mathbf{E}_k$ in slot $s_{i+1}$. We first define some helpful notation:

- $n_{rdy}$ $(\geq 0)$ is the number of stations which are ready to accept a new packet arrival during slot $s_i$. A station can accept a new packet if it is idle or receiving a broadcast of its own packet in slot $s_i$. Thus, $n_{rdy}$ is the number of stations whose states are either **ID** or $(\mathbf{ST}, i)$ in state $\mathbf{E}_j$.
- $n_{ar}$ $(0 \leq n_{ar} \leq n_{rdy})$ is the number of stations, among the $n_{rdy}$ stations ready to accept a new arrival, which actually have a new packet arrival during slot $s_i$. Thus, $n_{ar}$ is the number of the stations whose states change from **ID** or $(\mathbf{ST}, i)$ in slot $s_i$ to **IT** in slot $s_{i+1}$.
- $n_{tx}$ $(\geq 0)$ is the number of stations transmitting in slot $s_i$. This includes stations transmitting a new packet and stations retransmitting a packet. In other words, $n_{tx}$ is the number of stations whose state is either **IT** or $(\mathbf{RT}, i)$ in state $\mathbf{E}_j$. Note that one of

the $n_{tx}$ stations always succeeds with its transmission in slot $s_i$.

Using the above notation, the transition probability $p^i_{\mathbf{E}_j,\mathbf{E}_k}$ is calculated below. If there is no transmission in slot $s_i$, namely, if $n_{tx}$ is zero, stations are either idle (**ID**) or are waiting for feedback from their respective transmissions $((\mathbf{RT}, j)$ $(j \neq i)$ or $(\mathbf{ST}, j')$ $(j' \neq i))$. The only possible state changes in this case are changes from **ID** to **IT** caused by new arrivals at idle stations. Therefore, the state transition probability for this case becomes:

$$p^i_{\mathbf{E}_j,\mathbf{E}_k} = q^{n_{ar}}(1-q)^{n_{rdy}-n_{ar}}, \quad \text{for } n_{tx} = 0, \qquad (3-2)$$

where $q$ is the probability of having a new arrival at a station.

If there are $n_{tx}(> 0)$ stations transmitting in slot $s_i$, one of them will succeed with the probability $\frac{1}{n_{tx}}$ and change its state. From this and the above discussion to obtain eq.(3-2), we can obtain the state transition probability for this case as follows:

$$p^i_{\mathbf{E}_j,\mathbf{E}_k} = \frac{1}{n_{tx}} q^{n_{ar}}(1-q)^{n_{rdy}-n_{ar}}, \quad \text{for } n_{tx} > 0. \qquad (3-3)$$

Eqs.(3-2) and (3-3) give the slot-by-slot state transition probabilities from slot $s_i$ to slot $s_{i+1}$. The state transition probability matrix $P$ for transitions between imbedded points is given by:

$$P = \Pi^L_{i=1} P^i, \qquad (3-4)$$

where $P^i$ is the transition probability matrix from slot $s_i$ to slot $s_{i+1}$. The $(j,k)$-element of $P^i$ is $p^i_{\mathbf{E}_j,\mathbf{E}_k}$ given in eqs.(3-2) and (3-3). Note that imbedded points are at the slots belonging to channel 1, i.e., slots $s_1$ of the given cycle.

Let $\pi_{\mathbf{E}_j}$ be the steady state probability for a system to be in state $\mathbf{E}_j$ at imbedded points. The steady state distribution $\Pi = (\pi_{\mathbf{E}_1}, \ldots, \pi_{\mathbf{E}_{n_{state}}})$ is obtained by solving the following set of linear equations:

$$\begin{aligned} \Pi P &= \Pi, \\ \sum_j \pi_{\mathbf{E}_j} &= 1. \end{aligned} \qquad (3-5)$$

### 3.3 Throughput, Mean Delay, and Variance of Delay

From the steady state probability distribution $\Pi$ obtained in the previous subsection, we can obtain various performance measures. The throughput is obtained as the mean number of successful transmissions in slot $s_1$. Let $\mathcal{A}_1$ be a set of system states containing a station in state $(\mathbf{ST}, 1)$. If a station is in state $(\mathbf{ST}, 1)$ during slot $s_1$, it is receiving a successful broadcast of its own packet. Note that there is at most one station whose state is $(\mathbf{ST}, 1)$ in slot $s_1$. The throughput is then obtained as the mean number of stations in state $(\mathbf{ST}, 1)$ in $\mathcal{A}_1$. Thus, the throughput $T$ becomes:

$$T = \sum_{\mathbf{E}_j \in \mathcal{A}_1} 1 \times \pi_{\mathbf{E}_j}. \qquad (3-6)$$

Next, we obtain the transmission delay $D$. $D$ is given in eq.(2-1) as a function of $m$, the number of retransmissions required for a packet to be successfully transmitted. Its mean and variance are given in eqs.(2-2) and (2-3), respectively. Since stations are homogeneous, we focus on one station, say, test station, and obtain the delay distribution there without loss of generality.

Assume that the test station belongs to channel 1. We observe the system at imbedded points (i.e., slots $s_1$). Note that if the test station is in state $\mathbf{IT}$, it is transmitting a new packet. If the test station is in state $(\mathbf{ST}, 1)$, it is receiving a successful broadcast of its own packet. Assume that the test station first visits state $(\mathbf{ST}, 1)$ in $m+1$ transitions since it entered state $\mathbf{IT}$ last time. Then $m+1$ is the number of transmissions required for the test station to send a packet successfully. In other words, $m$ is the number of retransmissions required at the test station.

Now, let $p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$ denote the probability that the system visits state $\mathbf{E}_l$ for the first time in $m+1$ transitions, given the initial state is $\mathbf{E}_j$. $p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$ is numerically obtained by a simple calculation from $P$, the state transition probability matrix given in eq.(3-4). This procedure is summarized in Appendix A.

Using $p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$, we can obtain the mean and the variance of $m$, the number of retransmissions required for the test station to successfully transmit a packet, in the following way. Let $\mathcal{A}_2$ be a set of system states where the test station is in state $\mathbf{IT}$, or equivalently, a set of system states where the test station is transmitting a new packet. Let $\mathcal{A}_3$ be a set of system states where the test station is in state $(\mathbf{ST}, 1)$, or equivalently, a set of system states where the test station is receiving a successful broadcast of its packet. Then the mean and the variance of $m$ become:

$$E[m] = \frac{\sum_{\mathbf{E}_j \in \mathcal{A}_2} \pi_{\mathbf{E}_j} (\sum_{\mathbf{E}_l \in \mathcal{A}_3} \sum_{m=0}^{\infty} m p_{\mathbf{E}_j,\mathbf{E}_l}(m+1))}{\sum_{\mathbf{E}_j \in \mathcal{A}_2} \pi_{\mathbf{E}_j}}, \qquad (3-7)$$

$$Var[m] = \frac{\sum_{\mathbf{E}_j \in \mathcal{A}_2} \pi_{\mathbf{E}_j} (\sum_{\mathbf{E}_l \in \mathcal{A}_3} \sum_{m=0}^{\infty} m^2 p_{\mathbf{E}_j,\mathbf{E}_l}(m+1))}{\sum_{\mathbf{E}_j \in \mathcal{A}_2} \pi_{\mathbf{E}_j}} - (E[m])^2, \qquad (3-8)$$

where $\sum_{\mathbf{E}_j \in \mathcal{A}_i}$ is a summation taken over all system states in $\mathcal{A}_i$ $(i = 2, 3)$. In eq. (3-7), the term $\sum_{\mathbf{E}_l \in \mathcal{A}_3} \sum_{m=0}^{\infty} m p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$ is the mean number of retransmissions required at the test station, given that the initial system state is $\mathbf{E}_j$. Expectation is further taken over all the different possible initial system states $\mathbf{E}_j$. The second moment of $m$ is derived in a similar way, and the variance of $m$ is given in eq (3-8).

Substituting eqs.(3-7) and (3-8) into eqs.(2-2) and (2-3), we can obtain the mean and the variance of transmission delay $D$.

The analysis presented above is exact and holds for a slotted broadcast star network of any size. This analysis is, however, computationally practical only for networks with a small number of stations. For example, for a network with $L = \lfloor R \rfloor + 1 = 2$ and $N = 4$, our program requires 15.3 Mbytes memory space for double precision calculation. Therefore, in the next section, an approximate analysis is presented for a network having a large number of stations. The accuracy of approximation is examined through simulations in section 5.

## 4. Approximate Analysis of a Broadcast Star Network with a Finite Station Population

In order to proceed, we develop an approximate analysis for a large broadcast star network. We employ the following approximation to reduce the number of possible system states so that the computation involved becomes tractable.

**Uniform Access Approximation:**

- Suppose there are a certain number of busy stations in the system at the beginning of a slot in steady state. We assume that each of these busy stations accesses one of the next $L$ slots (including the current slot) with a uniform probability $\frac{1}{L}$.

In other words, we assume that the channel in which a busy station chooses to transmit is independent of the channel it previously accessed. This is an approximation, since transmissions from a station are confined to a particular channel, as we saw in section 2. Accuracy of this approximation will be examined through simulations in the numerical result section.

### 4.1 Conditional Moment of the Number of Retransmissions

As in section 3, we observe stations in channel 1 and obtain the performance measures there without loss of generality. We will assume that $R \geq 1$, and thus, the number of channels is $L \geq 2$. The case of $R < 1$ (i.e. $L = 1$) will be addressed at the end of this section.

Let $P_{n_1}(m)$ be the conditional probability that a packet (say, test packet) in channel 1 requires exactly $m$ more retransmissions (in addition to the retransmissions the test packet might have experienced so far) to be successfully transmitted, given that $n_1$ packets (including the test packet itself) access channel 1. Note that if the test packet is a new packet which arrived during the previous slot, $m$ is the total number of retransmissions required for the test packet to be successfully transmitted.

$P_{n_1}(m)$ can be obtained in the following way. When $n_1$ packets (including the test packet itself) are transmitted on the same channel, the probability that the test packet is selected for a broadcast at the switch is $\frac{1}{n_1}$. Therefore, $P_{n_1}(0)$, the probability that the test packet is successfully transmitted and suffers no more retransmissions, becomes:

$$P_{n_1}(0) = \frac{1}{n_1}. \qquad (4-1)$$

If the transmission fails (the probability of this is $1 - \frac{1}{n_1}$), the test packet is blocked at the switch and is retransmitted in channel 1 of the next cycle. If $k$ ($k = 0, 1, \cdots, N - n_1$) stations become busy in slot $s_L$ of the current cycle, then, $n_1 - 1$ packets (including the test packet) left over from channel 1 of the current cycle, along with $k$ new packets which arrived in slot $s_L$ of the current cycle, will access channel 1 of the next cycle. Therefore, the population of channel 1 of the next cycle becomes $n_1 - 1 + k$. The probability that the test packet goes through $m - 1$ additional retransmissions becomes $P_{n_1-1+k}(m - 1)$. Let $a_{n_1,k}$ be the conditional probability that $k$ stations become busy in slot $s_L$ of the current cycle, given that the population of channel 1 is $n_1$ at the beginning of the current cycle. We have the following recurrence relation:

$$P_{n_1}(m) = (1 - \frac{1}{n_1}) \sum_{k=0}^{N-n_1} a_{n_1,k} P_{n_1-1+k}(m - 1), \quad \text{for } m \geq 1. \qquad (4-2)$$

Using $P_{n_1}(m)$, we can obtain the moments of $m$, the number of retransmissions required for the test packet to be successfully transmitted. Let $M_{n_1}^l$ be the $l$-th conditional moment of $m$, given that $n_1$ packets, including the test packet, access the same slot. Namely,

$$M_{n_1}^l = \sum_{m=0}^{\infty} m^l P_{n_1}(m). \qquad (4-3)$$

From the definition, we have:

$$M_{n_1}^0 = 1 \quad (n_1 = 1, 2, \cdots, N). \qquad (4-4)$$

From eqs.(4-1) and (4-2), we obtain the following recurrence relation for $l \geq 1$ and $n_1 \geq 1$:

$$\begin{aligned} M_{n_1}^l &= P_{n_1}(0)0^l + \sum_{m=1}^{\infty} P_{n_1}(m)m^l = \sum_{m=1}^{\infty}(1 - \frac{1}{n_1}) \sum_{k=0}^{N-n_1} a_{n_1,k} P_{n_1-1+k}(m-1)m^l \\ &= (1 - \frac{1}{n_1}) \sum_{k=0}^{N-n_1} a_{n_1,k} (\sum_{r=0}^{l-1} \binom{l}{r} M_{n_1-1+k}^r + \binom{l}{l} M_{n_1-1+k}^l). \qquad (4-5) \end{aligned}$$

Now we will obtain $a_{n_1,k}$, an unknown term in eqs.(4-2) and (4-5). We first define events $Ar(k)$, $Ch1(n_1)$, and $Sys(n)$:

- $Ar(k)$ is the event where $k$ stations become busy during slot $s_L$ of the current cycle, or equivalently, the event where the arrival of a new packet occurs at $k$ idle stations during slot $s_L$ of the current cycle.
- $Ch1(n_1)$ is the event where the population of channel 1 is $n_1$ at the beginning of the current cycle.

- $Sys(n)$ is the event where the total number of busy stations is $n$ at the beginning of the current cycle.

Then, $a_{n_1,k}$, the conditional probability that $k$ stations become busy during slot $s_L$ of the current cycle, given that the population of channel 1 is $n_1$ at the beginning of the current cycle, is given by:

$$
\begin{aligned}
a_{n_1,k} &= \Pr[Ar(k)|Ch1(n_1)] \\
&= \sum_{n=n_1}^{N} \Pr[Ar(k) \wedge Sys(n)|Ch1(n_1)] = \sum_{n=n_1}^{N} \frac{\Pr[Ar(k) \wedge Sys(n) \wedge Ch1(n_1)]}{\Pr[Ch1(n_1)]} \\
&= \frac{1}{\Pr[Ch1(n_1)]} \sum_{n=n_1}^{N} \left(\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]\Pr[Sys(n) \wedge Ch1(n_1)]\right) \\
&= \frac{1}{\Pr[Ch1(n_1)]} \sum_{n=n_1}^{N} \left(\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]\Pr[Ch1(n_1)|Sys(n)]\Pr[Sys(n)]\right).
\end{aligned}
\tag{4-6}
$$

By noting that

$$\Pr[Ch1(n_1)] = \sum_{n=n_1}^{N} (\Pr[Ch1(n_1)|Sys(n)]\Pr[Sys(n)]), \tag{4-7}$$

the unknown terms in eq.(4-6) are $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$, $\Pr[Ch1(n_1)|Sys(n)]$, and $\Pr[Sys(n)]$. These unknown terms are obtained below.

$\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$ is the probability that $k$ stations become busy during slot $s_L$ of the current cycle, given that there are a total of $n$ busy stations at the beginning of the current cycle and that $n_1$ of these $n$ busy stations are transmitting in channel 1. Derivation of this term is summarized in Appendix B.

$\Pr[Ch1(n_1)|Sys(n)]$ is the probability that $n_1$ stations are transmitting in channel 1 of the current cycle, given that there are a total of $n$ busy stations in the system at the beginning of the current cycle. This probability can be easily obtained by using the Uniform Access Approximation, where a busy station is assumed to choose a slot randomly from the next $L$ slots and transmit in that slot:

$$\Pr[Ch1(n_1)|Sys(n)] = \binom{n}{n_1}\left(\frac{1}{L}\right)^{n_1}\left(1 - \frac{1}{L}\right)^{n-n_1}. \tag{4-8}$$

$\Pr[Sys(n)]$ is the probability that there are a total of $n$ busy stations in the system at the beginning of the current cycle. This probability is obtained below. Let $\mathcal{E}_n$ denote the state in which there are $n$ busy stations in the system at the beginning of a cycle, and let $\pi_n$ denote the steady state probability for a system to be in state $\mathcal{E}_n$. We first obtain the transition probability $p_{ij}$ for a system state to move from $\mathcal{E}_i$ to $\mathcal{E}_j$.

Assume that there are $i$ busy stations and $N-i$ idle stations in the system at the beginning of the current cycle. If $l$ ($l = 0, 1, \cdots, i$) stations out of $i$ busy stations become idle in the current cycle (call this probability $\alpha_i(l)$), there are $N-i+l$ idle stations which can potentially generate packets in the current cycle and thus become busy. If $k$ ($k = 0, 1, \cdots, N-i+l$) of these $N-i+l$ idle stations become busy in fact (let this probability be $\beta_{N-i+l}(k)$), there will be $i-l+k$ busy stations at the beginning of the next cycle. By taking a summation over all possible values of $l$ and $k$ with the condition $i-l+k=j$, we have the transition probability $p_{ij}$ from state $\mathbf{E}_i$ in the current cycle to $\mathbf{E}_j$ in the next cycle given by:

$$p_{ij} = \Pr[\mathbf{E}_j|\mathbf{E}_i] = \sum_{i-l+k=j, l\geq 0, k\geq 0} \alpha_i(l)\beta_{N-i+l}(k). \qquad (4-9)$$

The derivation of $\alpha_i(l)$ and $\beta_{N-i+l}(k)$ is presented in Appendix C.

The probability distribution of the number of busy stations at the beginning of a cycle in steady state, $\pi_n \equiv \Pr[Sys(n)]$, is obtained by numerically solving the following set of linear equations:

$$(\pi_0\pi_1\cdots\pi_N) = (\pi_0\pi_1\cdots\pi_N) \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0N} \\ p_{10} & p_{11} & \cdots & p_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N0} & p_{N1} & \cdots & p_{NN} \end{pmatrix} \qquad (4-10)$$

$$\sum_{i=0}^{N} \pi_i = 1. \qquad (4-11)$$

Since we have already obtained all the unknown terms from eq.(4-6), $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$, $\Pr[Ch1(n_1)|Sys(n)]$, and $\Pr[Sys(n)]$, we have completed the derivation of the probability $a_{n_1,k}$.

Once the probability $a_{n_1,k}$ is known, using eqs.(4-4) and (4-5), we can obtain $M_{n_1}^l$, the $l$-th conditional moment of $m$ for the given population of channel 1 ($n_1$). In the following subsection, we obtain the distribution for the given population $n_1$.

## 4.2 Distribution of the Number of Packets in a Channel

In the above subsection, we obtained the conditional moments of the number of required retransmissions $m$ for a given value of $n_1$. In this section, we observe slots immediately following the arrival of a new packet (test packet), and obtain the number of packets accessing those slots. Again, without loss of generality, we observe slots belonging to channel 1. Note that to obtain transmission delay $D$, we need the population size of channel 1 considering only those slots which follow the arrival of a new packet. We do not consider randomly chosen slots.

We note that the population of channel 1 in randomly chosen slots has already been obtained in eq.(4-7) as $\Pr[Ch1(n_1)]$. Here, let $\theta_{n_1}$ denote the same probability, i.e., $\theta_{n_1} \equiv \Pr[Ch1(n_1)]$. Using eqs.(4-7), (4-8) and (4-10), $\theta_{n_1}$ becomes:

$$\theta_{n_1} = \sum_{n=0}^{N} \binom{n}{n_1} (\frac{1}{L})^{n_1} (1 - \frac{1}{L})^{n-n_1} \pi_n. \qquad (4-12)$$

Now, we obtain the distribution of the population of channel 1 in slots which immediately follow the arrival of a new packet. Let us define $Nwtx$ as the following event:

- $Nwtx$ is the event where a packet arrives at a station (say, test station) during slot $s_L$ (the last slot of a cycle) and is accepted in its buffer.

Note that this test packet will be transmitted in slot $s_1$ of the next cycle. Note also that packets arriving at busy stations are lost. We observe the system in slots immediately following the event $Nwtx$. In other words, we observe the system only in slots where an initial transmission of a new packet occurs. These slots will be referred to as observation slots.

Let $\theta'_{n_1}$ denote the steady state probability that there are $n_1$ busy stations in observation slots. Remember that $Ch1(i)$ has been defined as the event where the population size of channel 1 is $i$, and that $Sys(n)$ has been defined as the event where $n$ stations are busy at the beginning of the current cycle. Let us define $Ch1'(n_1)$ as the following event:

- $Ch1'(n_1)$ is the event where $n_1$ stations (including the test station) transmit in $s_1$ of the next cycle.

Then $\theta'_{n_1}$ is given by:

$$\begin{aligned}\theta'_{n_1} &= \Pr[Ch1'(n_1)|Nwtx] = \frac{\Pr[Ch1'(n_1) \wedge Nwtx]}{\Pr[Nwtx]} \\ &= \frac{\sum_{i=0}^{n_1} \Pr[Ch1'(n_1) \wedge Nwtx \wedge Ch1(i)]}{\Pr[Nwtx]} \\ &= \frac{\sum_{i=0}^{n_1} \Pr[Nwtx|Ch1'(n_1) \wedge Ch1(i)] \Pr[Ch1'(n_1)|Ch1(i)]\Pr[Ch1(i)]}{\Pr[Nwtx]}. \qquad (4-13)\end{aligned}$$

$\Pr[Ch1(i)]$ in eq(4-13) is $\theta_i$ given in eq.(4-12). The other unknown terms in eq.(4-13) are obtained below.

$\Pr[Ch1'(n_1)|Ch1(i)]$ is the conditional probability that, given $i$ packets accessing slot $s_1$ of the current cycle, $n_1$ packets access slot $s_1$ of the next cycle. If $i \geq 1$, this probability is equal to the probability of having a new arrival at $n_1 - i + 1$ idle stations during slot $s_L$ of the current cycle. If $i = 0$, this is equal to the probability of having a new arrival at $n_1$ idle stations during slot $s_L$ of the current cycle. Therefore,

$$\Pr[Ch1'(n_1)|Ch1(i)] = \begin{cases} a_{i,n_1-i+1}, & \text{If } i \geq 1 \\ a_{0,n_1}, & \text{If } i = 0 \end{cases}, \qquad (4-14)$$

where $a_{i,k}$ is the probability that a new arrival occurs at $k$ idle stations during slot $s_L$ of the current cycle, given that the population size of channel 1 is $i$ at the beginning of the current cycle. $a_{i,k}$ has been obtained in subsection 4.1.

$\Pr[Nwtx|Ch1'(n_1) \wedge Ch1(i)]$ is the probability that a new packet arrives at the test station during slot $s_L$ of the current cycle, given that $i$ packets access slot $s_1$ of the current cycle, and that $n_1$ packets access slot $s_1$ of the next cycle.

Note that there are either $n_1 - i + 1$ (if $i \geq 1$) or $n_1$ (if $i = 0$) stations which have a new arrival during slot $s_L$ of the current cycle. Recall that there are a total of $N$ stations on the network. $\Pr[Nwtx|Ch1'(n_1) \wedge Ch1(i)]$, the probability that the test station is one of these stations having an arrival during slot $s_L$, is given by:

$$\Pr[Nwtx|Ch1'(n_1) \wedge Ch1(i)] = \begin{cases} \frac{n_1-i+1}{N}, & \text{If } i \geq 1 \\ \frac{n_1}{N}, & \text{If } i = 0 \end{cases}. \qquad (4-15)$$

The last unknown term in eq.(4-13), $\Pr[Nwtx]$ is obtained in the following way. Let $Sys_L(n)$ denote the following event:

- $Sys_L(n)$ is the event where there are $n$ busy stations in the system during slot $s_L$ of the current cycle.

$\Pr[Nwtx]$ becomes:

$$\Pr[Nwtx] = \sum_{n=0}^{N} \Pr[Nwtx \wedge Sys_L(n)] = \sum_{n=0}^{N} \Pr[Nwtx|Sys_L(n)]\Pr[Sys_L(n)]. \qquad (4-16)$$

$\Pr[Nwtx|Sys_L(n)]$ in the above equation is the probability that a new packet arrives at the test station during slot $s_L$ and is accepted in its buffer, given that there are $n$ busy stations in the system during slot $s_L$. A new packet arrives at the test station with the probability $q$. If there are a total of $n$ busy stations in the system, the probability that the test station is busy is $\frac{n}{N}$. Therefore, the probability that the test station is idle and can accept a new packet is $1 - \frac{n}{N}$. Thus, we have:

$$\Pr[Nwtx|Sys_L(n)] = q(1 - \frac{n}{N}). \qquad (4-17)$$

$\Pr[Sys_L(n)]$ in eq.(4-16) is the probability that there are $n$ busy stations in the system during slot $s_L$. Since all the channels are statistically identical in steady state, this probability is the same as $\pi_n$, the steady state probability that there are $n$ busy stations in the system during slot $s_1$. $\pi_n$ has been obtained in eqs. (4-10) and (4-11). Therefore, $\Pr[Nwtx]$ in eq.(4-13) becomes:

$$\Pr[Nwtx] = q(1 - \frac{n}{N})\pi_n. \qquad (4-18)$$

Substituting all the unknown terms obtained above into eq.(4-13), we can finally obtain $\theta'_{n_1}$ as follows:

$$\theta'_{n_1} = \frac{\frac{n_1}{N} a_{n_1,0}\theta_0 + \sum_{i=1}^{n_1} \frac{n_1-i+1}{N} a_{n_1-i+1,i}\theta_i}{\sum_{n=0}^{N} q(1-\frac{n}{N})\pi_n}. \qquad (4-19)$$

Once $\theta'_{n_1}$ is obtained, we can obtain the $l$-th moment of the number of retransmissions $m$ required for a new packet to be successfully transmitted by using values of $M^l_{n_1}$ given in eq.(4-5). The $l$-th moment becomes:

$$E[m^l] = E_{n_1}[M^l_{n_1}] = \sum_{n_1=0}^{N} M^l_{n_1}\theta'_{n_1}. \qquad (4-20)$$

Note that $M^l_{n_1}$ is the $l$-th conditional moment of $m$ for a given value of $n_1$. $E_{n_1}[\cdot]$ denotes the expectation over $n_1$.

From eqs.(4-20) and (2-1), we now know the statistics of the delay $D$. If the mean and variance of the delay is of primary interest, then they are given by eqs.(2-2) and (2-3) with the following equations:

$$E[m] = E_{n_1}[M^1_{n_1}], \quad Var[m] = E_{n_1}[M^2_{n_1}] - (E_{n_1}[M^1_{n_1}])^2. \qquad (4-21)$$

Note that given $n$ busy stations, the test station is busy with probability $\frac{n}{N}$. Further note that since packets arriving at a busy station are lost, the loss probability $P_{loss}$ due to buffer overflow becomes:

$$P_{loss} = \sum_{n=0}^{N} \frac{n}{N}\pi_n. \qquad (4-22)$$

Since the offered load to the system is $Nq$, we have the throughput $T$ as the following:

$$T = Nq(1 - P_{loss}).. \qquad (4-23)$$

Recall that, in the case of $R < 1$, there is only one logical channel. Thus, we need not employ the Uniform Access Approximation, and the analysis becomes exact. For the case of $R < 1$, eq.(4-6) is replaced by:

$$a_{n_1,k} = \binom{N-n_1}{k} q^k (1-q)^{N-n_1-k}. \qquad (4-24)$$

## 5. Numerical Results

In this section we show numerical results for a broadcast star network based on the analyses presented in sections 3 and 4. We also show simulation results to examine the accuracy of the approximation made in section 4. We present the results graphically with the unit of time equal to one slot.

### 5.1 Finite Station Population: Small System

For a broadcast star network with a small station population, the throughput, the mean delay, and the variance of delay can be obtained through the exact analysis presented in section 3. However, due to the computational limitations associated with the exact analysis, the broadcast star we examine in this subsection is rather small. It has a maximum of 4 stations. The propagation delay $R$ is assumed to be 1.0, thus the number of channels $L = \lfloor R \rfloor + 1$ is 2. $R$ denotes the round trip propagation time between a station and the switch (measured in slots). For instance, $R$ equals 5.0 with a channel speed of 1G bits/sec and a packet length of 1000 bits, if the round trip distance between a station and the switch is 1000 meters, and the signal propagation speed is $2 \times 10^8$ m/sec.

Figures 2, 3, and 4 show the throughput $T$, the mean delay $E[D]$, and the variance of delay $Var[D]$ for various values of $N$ (the number of stations) as a function of $q$, the probability of a new packet arrival at a station. Along with the values from the exact analysis presented in section 3, the results from the approximate analysis in section 4 are shown in these figures. Exact results and approximations are shown by solid lines and dashed lines, respectively. (Note: since the variance of delay equals zero in the case of $N = 1$, it is not shown in Figure 4.)

The following can be seen from the throughput values obtained through the exact analysis in Figure 2. In the cases of $N = 1$ and 2, the throughput increases slowly to the maximum values as $q$ increases. The maximum throughput for the case of $N = 1$ is $\frac{1}{3}$. This can easily be seen from the following discussion. Since there is only one station in the system, each transmission always succeeds. Assume that a station transmits in slot 1. The transmission is successful, and the station receives a broadcast of its packet in slot 2. If a new packet arrives at a station during slot 2, it will be lost. (This follows from our preliminary assumptions.) A new arrival in slot 3 is accepted and is transmitted in slot 4. This transmission succeeds. Therefore, under a heavy traffic situation, there is a successful transmission in every third slot, and the throughput becomes $\frac{1}{3}$. The maximum throughput in the case of $N = 2$ is $\frac{2}{3}$. This is because each of the two stations contributes $\frac{1}{3}$ to the total throughput. In the case of $N = 3$, the throughput increases rather rapidly to the maximum throughput 1.0 as $q$

increases. In such a case, all the channels are completely utilized under heavy traffic. In the case of $N = 4$, the throughput increases rather rapidly until $q$ becomes 0.5 and then slowly rises to the maximum throughput 1.0 as $q$ increases.

In Figure 2, values from the approximation analysis closely agree with those from the exact analysis at all the values of $q$, when $N$ is 1 and 2. For $N = 3$ and 4, our approximation is good in the area where $q$ is smaller than 0.6. When $q$ is larger than 0.6, our approximation underestimates the throughput. We conjecture an explanation of this difference as follows. When traffic is heavy, it is very likely that there is a new packet arrival at a station immediately after the completion of the current transmission. This introduces a strong co-relation between the channel that a station currently belongs to and the channel that the station will use next. This co-relation among the channels breaks the Uniform Access Approximation made in section 4. However, as we will see later in Figures 5, 6, and 7, our approximations becomes more accurate when a network is larger.

Figures 3 and 4 show the mean and the variance of delay as a function of $q$ for various values of $N$, the number of stations. In Figure 3, the delay in the case of $N = 1$ is a constant 2.0, because there is no contention. In cases of $N = 2$ and 3, the mean delay slightly increases up to a certain maximum value and then decreases as $q$ increases. In Figure 4, the same trend can be seen. The variance of delay first increases, then starts decreasing as $q$ increases for the $N = 2, 3$ cases. This decrease in the mean and the variance of delay can be explained by "regularity" of transmissions which appear in a heavy traffic situation. As an example, let's consider a network with 3 stations. Consider an extreme case where $q = 1$, the case in which a new packet arrives at each station during every slot. As shown in the example used in Figure 2, once a station succeeds in transmission, its transmissions are confined to a sequence of slots commencing with the slot where the first successful transmission occurs. Transmission by this station continues with every third slot thereafter unless some other station seizes the channel. Therefore, once the three stations have successful transmissions in three successive slots (i.e., station $a$ succeeds in slot $i$, station $b$ in slot $i + 1$, station $c$ in slot $i + 2$), stations are locked into this situation, and each station continues to succeed in every third slot. Because of this regularity in transmissions, the mean and the variance of delay decreases as $q$ continues to increase beyond some value.

Figures 3 and 4 show that the mean and the variance of delay from the approximation agree very well with those from the exact analysis.

### 5.2 Finite Station Population: Large System

Due to the computational limitations of the exact analysis, we use the approximate analysis given in section 4 and investigate the performance of a broadcast star with a large station population. Note the analysis in section 4 is exact in the case of $R < 1.0$ and is approximate in the case of $R \geq 1$.

Figures 5, 6, and 7 show the throughput $T$, the mean delay $E[D]$, and the variance of delay $Var[D]$ as a function of the arrival probability $q$ for various values of $N$. $R$ is assumed to be 5.0 in these figures. Results from the approximate analysis are shown as real lines.

Simulation results are also shown and are indicated by symbols.

The following can be seen from the simulated throughputs in Figure 5. For a given value of $N$, the throughput first rapidly increases, and then slowly approaches the maximum throughput of 1.0, as $q$ increases. As the number of stations $N$ becomes larger, the throughput becomes larger for the same arrival probability $q$. This is because the total input to the network ($Nq$) is larger for a given value of $q$, if the value of $N$ is larger.

In Figure 5, the throughput values from the approximate analysis agree well with the simulation results when the value of $N$ is large. Even when $N$ is small, values from the approximation and simulations agree well except for heavy traffic situations (i.e., the area where the throughput value is larger than 0.8.). The reason for this difference is explained in the previous subsection. (See the discussion for Figure 2).

In Figure 6, the mean delay rapidly increases as $q$ increases, and then levels off. This is due to the fact that the system is a loss system where packets arriving at stations which already have packets are simply lost. In Figure 7, the variance of delay rapidly increases and stays at a certain maximum value as $q$ increases.

Figures 6 and 7 show that the mean and the variance of delay given by the approximations agree very well with the simulation results over the whole range of possible $q$.

90% confidence intervals for some of the simulation results used in Figures 6 and 7 are shown in Table 1. It can be easily seen that confidence intervals for the simulation results are very small. This is true with the simulation results in the other figures, and thus confidence intervals are not shown in the figures.

Figure 8 shows the mean delay $E[D]$ as a function of the throughput $T$ for the same networks in Figures 5 to 7. Simulation results are also shown and are indicated by symbols. In addition, for comparison purposes, the mean delay for an infinite population case is obtained from the analysis presented in the companion paper [11] and is shown in this figure. The only difference between an infinite population and a finite population case in this figure is the number of stations. All the other parameter values are kept the same.

In Figure 8, the mean delay stays small until the throughput becomes 0.8. Beyond that point, throughput increases rapidly to $N$ in the finite population case, and to infinity in the infinite population case. For a given value of throughput, as the number of stations $N$ increases (in the finite population case), the mean delay increases and approaches the delay of the infinite station population case.

## 6. Conclusion

In this paper, we analyzed the performance of a new network architecture based on collision avoidance: the broadcast star network. This network has the potential of combining the benefits of random access (low delay when traffic is light; simple, distributed, and therefore robust protocols) with excellent network utilization. We developed an exact analysis for a network with a finite station population. Due to the computationally demanding nature of the exact analysis, this analysis is practical only for networks with a small number of stations. Therefore, an approximate analysis is also presented for a network having a large number of stations. Through simulations, we show that the approximate analysis gives accurate estimates of the throughput, the mean, and the variance of transmission delay.

Through numerical results, we show that a collision avoidance broadcast star network obtains the maximum throughput of 1. Numerical results also show that the broadcast star network provides small transmission delays where the throughput is smaller than 0.8, even when the propagation delay between stations and the switch is long. Therefore, a broadcast star network is suitable for high speed networking environments in which the propagation delay on a transmission line becomes large compared to packet transmission time.

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## Appendix A. Derivation of $p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$

$p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$ denotes the probability that the system visits $\mathbf{E}_l$ for the first time in $m+1$ transitions, given the initial state is $\mathbf{E}_j$. This probability is numerically obtained from $P$, the state transition probability matrix given in eq.(3-4), by following the algorithm described below:

1. Observe that the $(j,l)$-element of the transition probability matrix $P$ is $p_{\mathbf{E}_j,\mathbf{E}_l}(1)$.
2. Replace the $(j,l)$-element of $P$ with zero. This eliminates the one step transition from $\mathbf{E}_j$ to $\mathbf{E}_l$. Let $P_1$ denote the resulting matrix.
3. Calculate $P_{k-1}P$ $(k = 2, 3, \cdots, \infty)$, and observe that the $(j,l)$-element of $P_{k-1}P$ is $p_{\mathbf{E}_j,\mathbf{E}_l}(k)$. Replace the $(j,l)$-element of $P_{k-1}P$ with zero, and let $P_k$ denote the resulting matrix.

Then, the $(j,l)$-element of matrix $P_mP$ gives the probability $p_{\mathbf{E}_j,\mathbf{E}_l}(m+1)$.

## Appendix B. Derivation of $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$

$\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$ is the probability that the arrival of a new packet occurs at $k$ idle stations during slot $s_L$ of the current cycle, given that the population of channel 1 at the beginning of the current cycle is $n_1$ and that the total number of busy stations at the beginning of the current cycle is $n$.

In order to obtain $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$, we first consider the stations which can contribute to the traffic in slot $s_L$ of the current cycle. They are:

- stations which are idle with no packet to send at the beginning of the current cycle (there are $N - n$ such stations, if $n$ stations are busy at the beginning of the current cycle), and
- stations which become idle in slots from $s_2$ through $s_L$ of the current cycle.

Note that if we consider channels 2 through $L$ of the previous cycle and we count the number of busy channels, we have also counted the number of stations which will change to an idle state in slots from $s_2$ through $s_L$ of the current cycle. This can be easily seen from the following discussion. A station transmitting a packet finds the outcome of its transmission attempt $R$ slots later, and once it finds the transmission successful, it becomes idle and ready to generate a new packet. Therefore, stations which succeeded in transmission in slots from $s_2$ through $s_L$ in the previous cycle can become idle and ready to generate a new packet in slots from $s_2$ through $s_L$ of the current cycle. The number of such stations is equal to that of busy channels in channels from 2 to $L$ in the previous cycle, since there is always one successful transmission in a busy channel.

$\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$ can be obtained in the following way. Assume that the population of channel 1 at the beginning of the current cycle is $n_1$ (event $Ch1(n_1)$) and that the total number of busy stations in the system at the beginning of the current cycle is $n$ (event $Sys(n)$). Assume that the arrival of a new packet occurs at $k$ idle stations during slot $s_L$ of the current cycle (event $Ar(k)$). Assume further that $k'$ out of these $k$ stations are among those which become idle during the current cycle (from slots $s_2$ through $s_L$), and that the remaining $k - k'$ stations are among those which are idle at the beginning of the current cycle. Let us define $Tsuc(l)$, $Ar1(k', l)$ and $Ar2(k - k', N - n)$ as the following events:

- $Tsuc(l)$ is the event where $l$ $(0 \leq l \leq L-1)$ stations become idle during slots from $s_2$ to $s_L$ of the current cycle, or equivalently, an event where there are $l$ successful transmissions in channels 2 to $L$ of the previous cycle. Note that if there are $n$ busy stations at the beginning of the current cycle and if there are $n_1$ busy stations during slot $s_1$ of the current cycle, there were $n - n_1$ busy stations during slots $s_2$ through $s_L$ of the previous cycle. Therefore, $l$ is less than or equal to $n - n_1$.
- $Ar1(k', l)$ is the event where $k'$ $(0 \leq k' \leq l)$ stations among the $l$ stations which become idle in the current cycle generate a new packet in slot $s_L$ of the current cycle.
- $Ar2(k - k', N - n)$ is the event where, among the $N - n$ idle stations at the beginning of the current cycle, $k - k'$ $(0 \leq k - k' \leq N - n)$ stations generate a new packet in slot

$s_L$ of the current cycle.

Then, $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$ is given by the following:

$$\begin{aligned}
&\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)] \\
&= \sum_{0 \le k-k' \le N-n, k \ge 0, k' \ge 0} (\Pr[Ar2(k-k', N-n)] \\
&\times \sum_{k' \le l \le min(L-1, n-n_1)} \Pr[Ar1(k', l) \mid Tsuc(l)\ ] \times \Pr[Tsuc(l) \mid Sys(n) \wedge Ch1(n_1)]). \qquad (B-1)
\end{aligned}$$

In the above equation, summations are taken over all possible values of $k'$ and $l$.

Probabilities which appear on the right hand side of eq.(B-1) are obtained below. $\Pr[Ar2(k-k', N-n)]$ is the probability that, among $N-n$ idle stations at the beginning of a cycle, $k-k'$ stations have a new packet arrival during slot $s_L$. Since the probability that a station does not have any arrivals from slots $s_1$ through $s_{L-1}$ but has an arrival during slot $s_L$ is $(1-q)^{L-1}q$, the first term on the right hand side of eq.(B-1) becomes:

$$\Pr[Ar2(k-k', N-n)\ ] = \binom{N-n}{k-k'}((1-q)^{L-1}q)^{k-k'} \times (1-(1-q)^{L-1}q)^{N-n-(k-k')}. \qquad (B-2)$$

The second term in eq.(B-1), $\Pr[Ar1(k', l) \mid Tsuc(l)\ ]$, is the conditional probability that $k'$ stations among $l$ stations generate a new packet during slot $s_L$ of the current cycle, given that these $l$ stations became idle during some slots $s_2$ through $s_L$ of the current cycle. Using the Uniform Access Approximation described at the beginning of section 4, these $l$ stations are uniformly distributed over slots $s_2$ through $s_L$. There are $\binom{L-1}{l}$ possible ways to distribute these $l$ stations over $L-1$ slots. If one of these $l$ stations becomes idle in the middle of the current cycle, say, in slot $s_{b_i}$ $(2 \le b_i \le L,\ 1 \le i \le l)$, the probability that there is no packet arrival during slots $s_{b_i}$ through $s_{L-1}$ but there is a packet arrival during slot $s_L$ at that station is $q(1-q)^{L-b_i}$. Therefore, the probability that $k'$ stations (among $l$ stations) have an arrival only during slot $s_L$ is $\prod_{1 \le i \le k'}(q(1-q)^{L-b_i}) \prod_{k' < i \le l}(1-q(1-q)^{L-b_i})$. By taking a summation over all possible ways to distribute $k'$ stations and $k-k'$ stations among slots $s_2$ through $s_L$ in such a way that no more than one station falls in one slot, $\Pr[Ar1(k', l) \mid Tsuc(l)\ ]$ becomes:

$$\Pr[Ar1(k', l) \mid Tsuc(l)\ ] = \frac{1}{\binom{L-1}{l}} \sum_{b_i \in B_1} \prod_{1 \le i \le k'} (q(1-q)^{L-b_i}) \prod_{k' < i \le l} (1-q(1-q)^{L-b_i}),$$

$$(B-3)$$

where $\mathcal{B}_1$ is a set of $l$ integers $b_i$ $(i = 1, 2, ..., l)$ which satisfy

$$\mathcal{B}_1 = \{\text{integers } b_i : i = 1, 2, \cdots, l, b_i \neq b_j (\text{if } i \neq j),\\ 2 \leq b_1 < b_2 < \cdots < b_{k'} \leq L, 2 \leq b_{k'+1} < b_{k'+2} < \cdots \leq b_l \leq L\}.$$

Note that $\sum_{b_i \in \mathcal{B}_1}$ in eq.(B-3) takes a summation over all possible ways to distribute $k'$ stations and $k - k'$ stations among slots $s_2$ through $s_L$.

The last term on the right hand side of eq.(B-1), $\Pr[Tsuc(l)|Sys(n) \wedge Ch1(n_1)]$, is obtained in the following way. Assume that the total number of busy stations in the system at the beginning of the current cycle is $n$ (event $Sys(n)$) and that $n_1$ out of these $n$ busy stations belong to channel 1 (event $Ch1(n_1)$). The remaining $n - n_1$ busy stations belong to channels other than 1. There are $(L-1)^{n-n_1}$ possible ways to distribute these $n - n_1$ busy stations over $L-1$ channels. Among these $(L-1)^{n-n_1}$ possible ways, there are $\binom{L-1}{l} g_{n-n_1,l}$ ways to distribute $n - n_1$ busy stations among $L - 1$ channels in such a way that there are only $l$ busy channels (among $L-1$ channels). Here, $\binom{L-1}{l}$ is the number of possible ways to choose $l$ channels from $L - 1$ channels, and for a given value of $l$, $g_{n-n_1,l}$ is the number of possible ways to distribute $n - n_1$ busy stations among $l$ channels in such a way that each of the $l$ channels contains at least one busy station. $g_{n-n_1,l}$ is obtained by solving the following recurrence relation:

$$g_{i,1} = 1, \qquad (B-4)$$

$$g_{i,l} = \sum_{s=1}^{l} \binom{l}{s} g_{i,s} - \sum_{s=1}^{l-1} \binom{l}{s} g_{i,s}. \qquad (B-5)$$

Note that $\sum_{s=1}^{l} \binom{l}{s} g_{i,s}$ is the number of all possible ways to distribute $i$ stations over $s$ channels, and thus, is equal to $l^i$. From the above discussion,

$$\Pr[Tsuc(l)|Sys(n) \wedge Ch1(n_1)] = \frac{\binom{L-1}{l} g_{n-n_1,l}}{(L-1)^{n-n_1}}. \qquad (B-6)$$

Finally, $\Pr[Ar(k)|Sys(n) \wedge Ch1(n_1)]$ can be obtained by substituting eqs.(B-2), (B-3) and (B-6) into eq.(B-1).

### Appendix C. Derivation of $p_{ij}$ in eq. (4-9)

In this appendix, we obtain probabilities $\alpha_i(l)$ and $\beta_{N-i+l}(k)$, which appear in eq. (4-9). $\alpha_i(l)$ is the conditional probability that, given $i$ stations are busy at the beginning of the current cycle, $l$ stations out of $i$ busy stations become idle in the current cycle. There are $L^i$ possible ways to distribute $i$ busy stations over $L$ channels. Among these $L^i$ possible ways, there are $\binom{L}{l}g_{i,l}$ ways to distribute $i$ busy stations among $L$ channels in such a way that there are only $l$ busy channels (among $L$ channels). Here, $\binom{L}{l}$ is the number of possible ways to choose $l$ channels from $L$ channels, and for a given value of $l$, $g_{i,l}$ is the number of possible ways to distribute $i$ busy stations among $l$ channels in such a way that each of the $l$ channels contains at least one busy station. $g_{i,l}$ is obtained by solving the recurrence relation given by eqs.(B-4) and (B-5). Therefore,

$$\alpha_i(l) = \frac{\binom{L}{l}g_{i,l}}{L^i}. \qquad (C-1)$$

$\beta_{N-i+l}(k)$ is the probability that $k$ out of $N-i+l$ idle stations become busy in the current cycle, given that there are $i$ busy stations at the beginning of the current cycle and that $l$ out of these $i$ stations become idle in the current cycle. Assume that $k'$ out of $k$ stations are among the $l$ stations which become busy in the current cycle (let $\gamma_l(k')$ denote this probability), and that the remaining $k-k'$ stations are among the $N-i$ stations which are idle at the beginning of the current cycle (let $\delta_i(k-k')$ denote this probability). Note that $k' = 0, 1, \cdots, k$. Then, $\beta_{N-i+l}(k)$ is given by:

$$\beta_{N-i+l}(k) = \sum_{0 \le k' \le k} \gamma_l(k')\delta_i(k-k'). \qquad (C-2)$$

$\gamma_l(k')$ in the above equation is the conditional probability that $k'$ stations generate a new packet in the current cycle, given that these $k'$ stations are among the $l$ stations which become idle in the current cycle. Using the Uniform Access Approximation, these $l$ stations are uniformly distributed over slots $s_1$ through $s_L$. Therefore, there are $\binom{L}{l}$ possible ways to distribute these $l$ stations over $L$ slots. If a station (among these $l$ stations) becomes idle in the middle of the current cycle, say, in slot $s_{b_i}$ $(1 \le b_i \le L,\ 1 \le i \le l)$, the probability that there is no packet arrival during some slot $s_{b_i}$ through $s_L$ at that station is $(1-q)^{L+1-b_i}$, and the probability that the station becomes busy is $1-(1-q)^{L+1-b_i}$. Therefore, the probability that $k'$ stations (among $l$ stations) have an arrival in the current cycle is $\prod_{1 \le i \le k'}(1-q)^{L+1-b_i}\prod_{k'<i\le l}(1-(1-q)^{L+1-b_i})$. By taking a summation over all possible ways to distribute $k'$ stations and $k-k'$ stations among slots $s_1$ through $s_L$ in such a way that no more than one station falls in one slot, $\gamma_l(k')$ becomes,

$$\gamma_l(k') = \frac{1}{\binom{L}{l}} \sum_{b_i \in B_2} \{ \prod_{1 \le i \le k'} (1-q)^{L+1-b_i} \prod_{k'<i\le l} (1-(1-q)^{L+1-b_i})\}, \qquad (C-3)$$

where

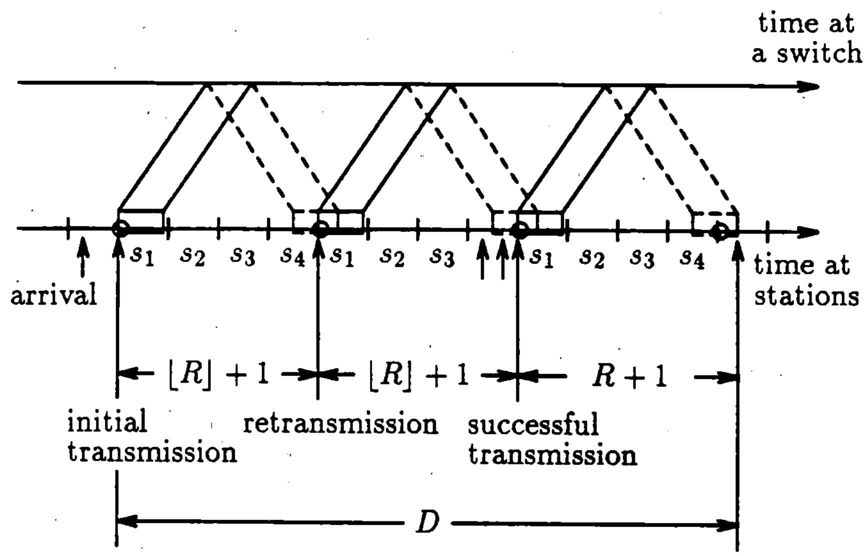
$$\mathcal{B}_2 = \{\text{integers } b_i, i = 1, 2, \cdots, l, b_i \neq b_j (\text{if } i \neq j),$$
$$1 \leq b_1 < b_2 < \cdots < b_{k'} \leq L, 1 \leq b_{k'+1} < b_{k'+2} < \cdots < b_l \leq L\}.$$

Note the summation $\sum_{b_i \in \mathcal{B}_2}$ is taken over all possible patterns of distributing $k'$ stations and $l - k'$ stations in such a way that no more than one station falls in one slot.

$\delta_i(k - k')$ in eq.(C-2) is the conditional probability that, given $N - i$ stations are idle at the beginning of the current cycle, $k - k'$ of $N - i$ stations remain idle at the end of the current cycle. This probability is obtained as follows. The probability that an idle station (at the beginning of the current cycle) remains idle until the end of the current cycle is $(1 - q)^L$. Therefore, the probability that an idle station becomes active during the current cycle is $1 - (1 - q)^L$. There are $\binom{N-i}{k-k'}$ ways to choose $k - k'$ stations from $N - i$ stations which are idle at the beginning of the current cycle. Therefore, $\delta_i(k - k')$ is given by:

$$\delta_i(k - k') = \binom{N - i}{k - k'} (1 - (1 - q)^L)^{k-k'} ((1 - q)^L)^{N-i-(k-k')}. \qquad (C - 4)$$

Substituting eqs.(C-3) and (C-4) into eq.(C-2), we can obtain $\beta_{N-i+l}(k)$. Once we obtain $\alpha_i(l)$ (eq.(C-1)) and $\beta_{N-i+l}(k)$, we can obtain $p_{ij}$ from eq.(4-9).




$s_i$ : slot belongs to channel $i$ ($i = 1, 2, 3, 4$)

Fig.1 Example of Packet Transmission in a Broadcast Star ($R = 3.5$)

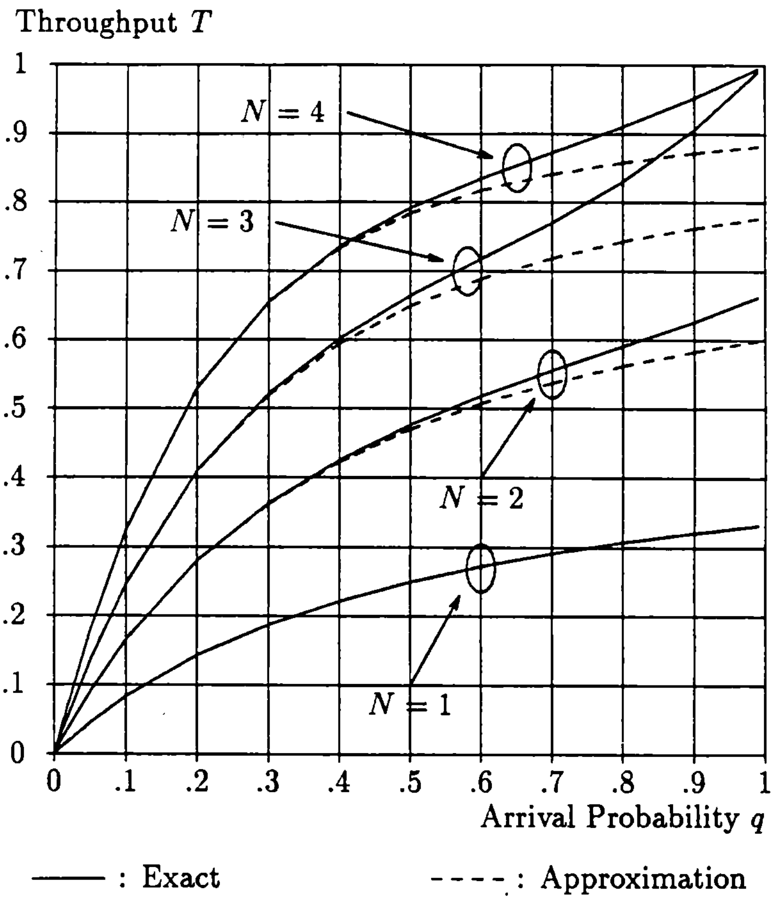



Fig.2 Throughput in a Small Broadcast Star
($R = 1,\ N = 1, 2, 3, 4$)

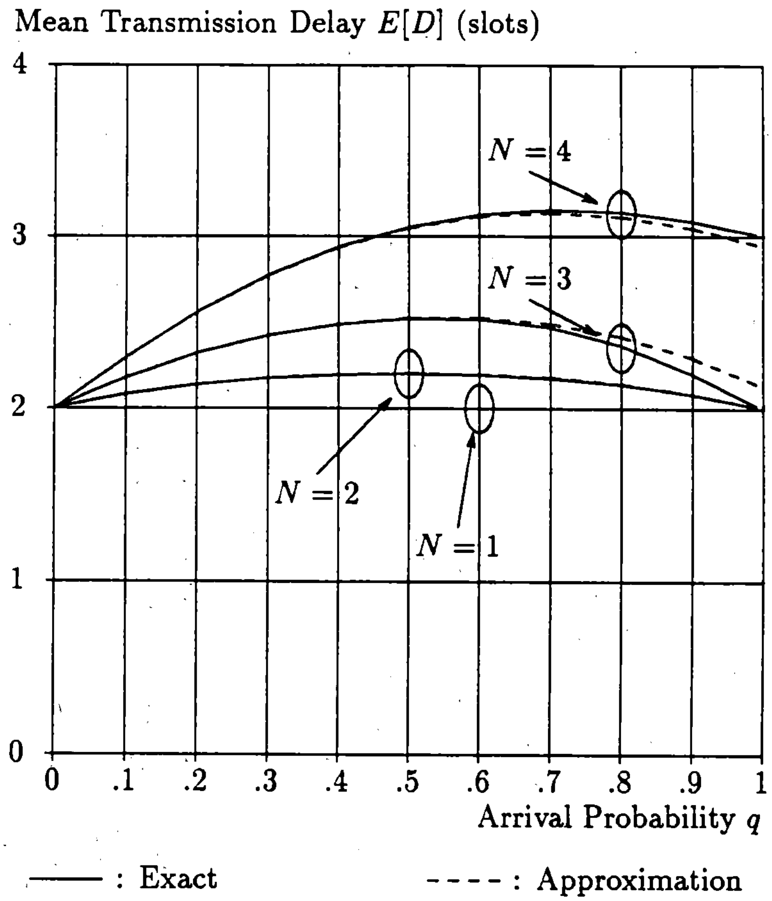



Fig.3 Mean Transmission Delay in a Small Broadcast Star
($R = 1,\ N = 1, 2, 3, 4$)

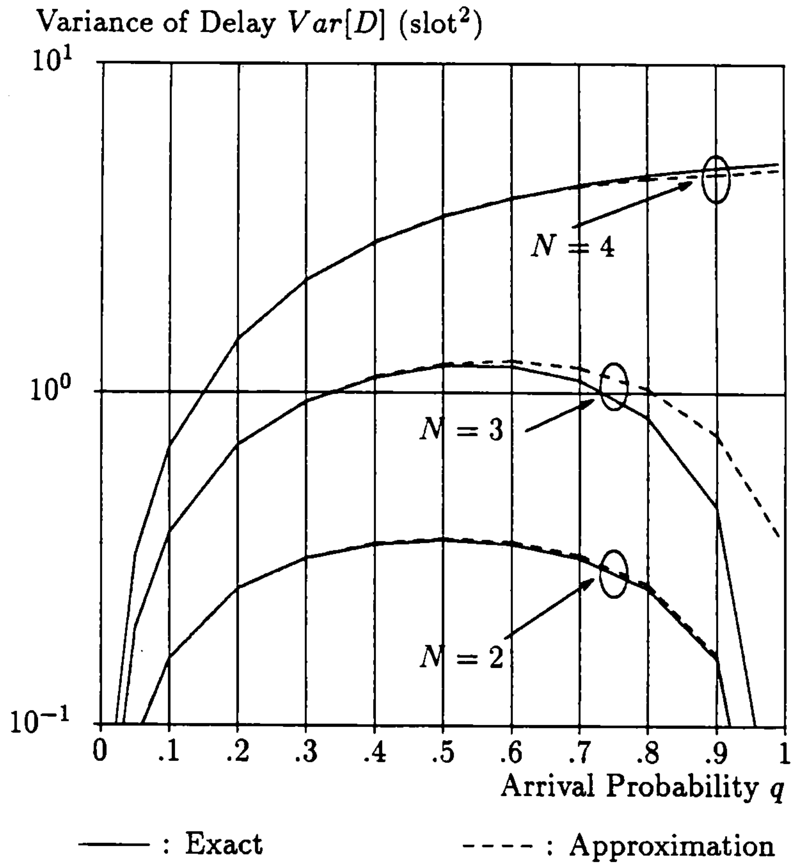
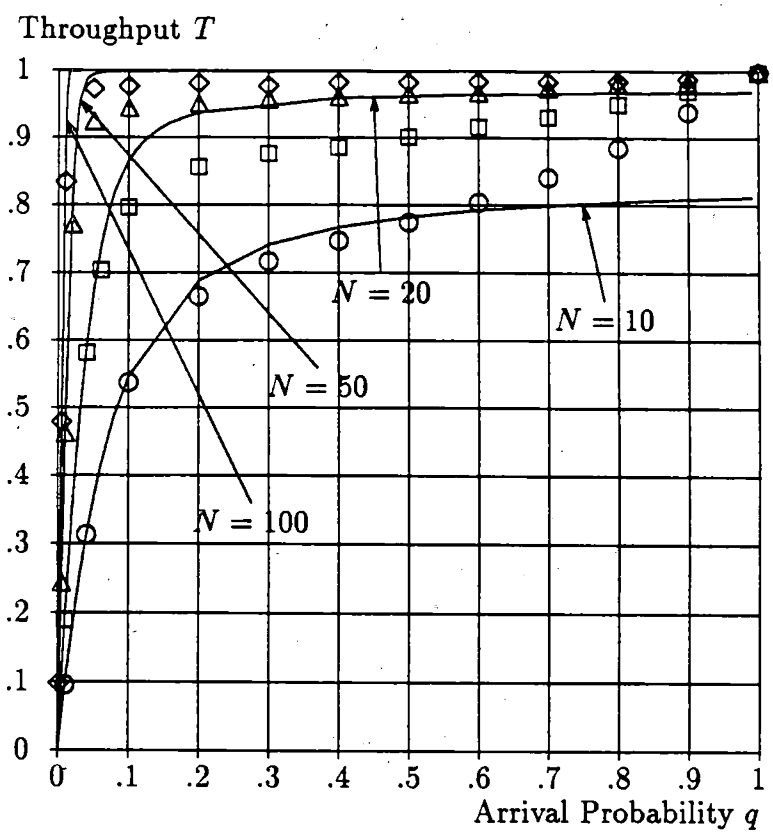



Fig.4 Variance of Transmission Delay in a Small Broadcast Star ($R = 1$, $N = 1, 2, 3, 4$)

Table 1. Simulation Results with 90% Confidence Intervals
(Slotted Broadcast Star with Finite Station Population, $N = 100$, $R = 5$)

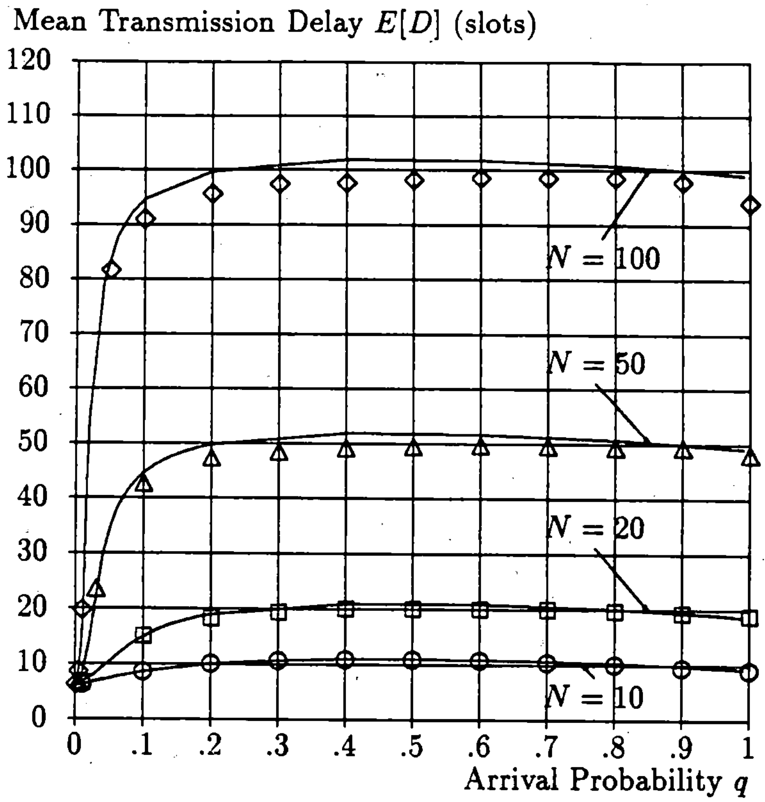
| $q$ | Mean delay | Variance of delay |
|---|---|---|
| 0.001 | 6.348 ±0.02879 | 2.458 ±0.29540 |
| 0.010 | 19.729 ±0.28503 | 561.889 ±23.7824 |
| 0.020 | 54.149 ±0.43573 | 6042.476 ±509.065 |
| 0.900 | 97.860 ±0.16871 | 44418.495 ±5237.66 |




Simulations

○: N=10 □: N=20 △: N=50 ◇: N=100

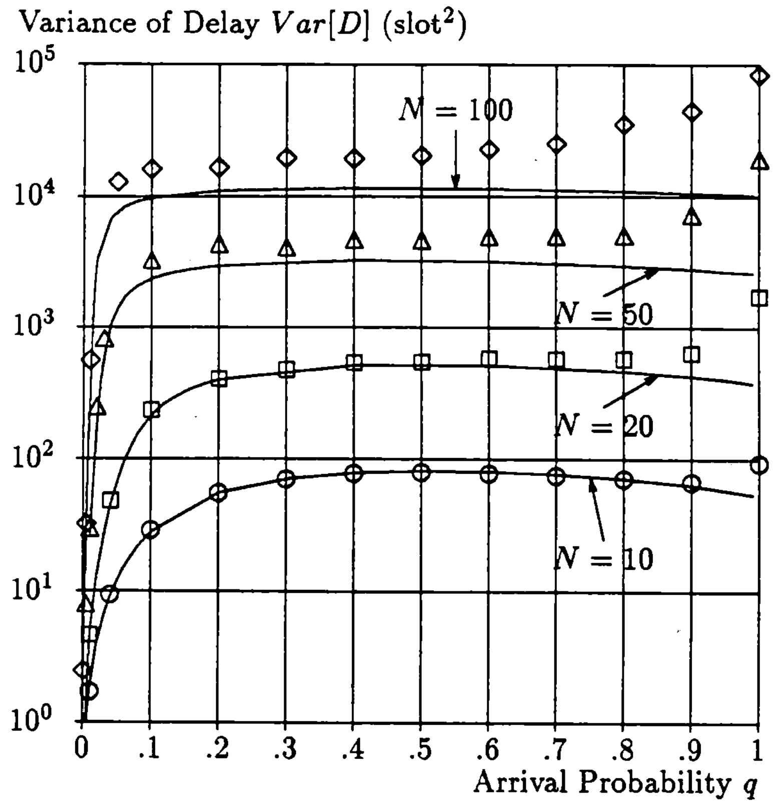
Fig.5 Throughput in a Large Broadcast Star with Finite Station Population ($R = 5$, $N = 10, 20, 50, 100$)




Simulations

○: N=10 □: N=20 △: N=50 ◇: N=100

Fig.6 Mean Transmission Delay in a Large Broadcast Star with Finite Station Population ($R = 5$, $N = 10, 20, 50, 100$)




Simulations

○: N=10 □: N=20 △: N=50 ◇: N=100

Fig.7 Variance of Transmission Delay in a Large Broadcast Star with Finite Station Population ($R = 5$, $N = 10, 20, 50, 100$)

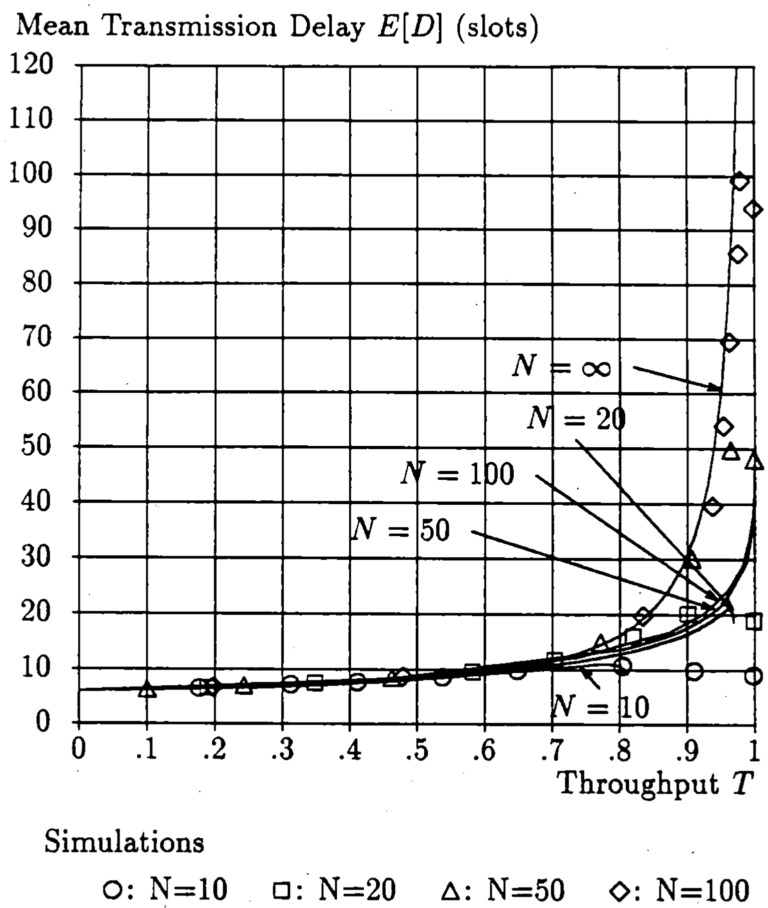



Fig.8 Throughput vs. Delay in a Large Broadcast Star with Finite Station Population ($R = 5$, $N = 10, 20, 50, 100$)